# W3L1 - METHOD OF UNDETERMINED COEFFICIENTS

Consider the equation

$$ay'' + by' + cy = g(t) \qquad \text{Eq. 1}$$

We look for a solution of the form:

$$y_h + y_p$$

where $y_h$ is the general solution to

$$ay'' + by' + cy = 0 \qquad \text{Eq. 2}$$

and $y_p$ is a particular solution to Eq. 1.

Claim: If $y_h$ solves (2) and $y_p$ solves (1), then $y = y_h + y_p$ also solves Eq. 1.

Proof: $y = y_h + y_p \Rightarrow y' = y_h' + y_p' \Rightarrow y'' = y_h'' + y_p''$

Plug into (1):

$$a(y_h'' + y_p'') + b(y_h' + y_p') + c(y_h + y_p)$$

Regroup:

$$= (a y_h'' + b y_h' + c y_h) + (a y_p'' + b y_p' + c y_p)$$

since $y_h$ solves (2)

$$= 0 + g(t) \qquad = g(t) \qquad \square$$

Note: $y(t) = y_h + y_p$ is a general solution, since $y_h$ depends on the two constants as in the last section.

Question: How do we find particular solutions?
2 methods will be covered:

1. Method of Undetermined Coefficients (Guessing)
2. Variation of Parameters (Better guessing)

## METHOD OF UNDETERMINED COEFFICIENTS
(Aka educated guess method)

EX: $y'' + 3y' + 2y = 3t$

- We need a function where $y$ ends up as a linear function after having its zeroth, first and second derivatives combined.
- Try a linear function

Let $y_1(t) = At$ (Guess !)

Check if such an $A$ exists so that $y_1$ solves the DE

$$y_1'' + 3y_1' + 2y_1 = 0 + 3(A) + 2(At) \overset{\text{set}}{=} 3t$$

$y_1' = A$
$y_1'' = 0$

constant

$$3A + 2At = 3t$$

$$3A = 0 \qquad 2A = 1$$

$$A = 0 \text{ and } A = 3/2$$

Not possible!

Try, $y_2(t) = At + B$ (Better guess)

$$y_2'' + 3y_2' + 2y_2 = 0 + 3(A) + 2(At + B) \overset{\text{set}}{=} 3t$$

$y_2' = A$
$y_2'' = 0$

Group $t$'s

$$2At + (3A + 2B) = 3t$$

$$3A + 2B = 0$$

$$2A = 3$$

$$A = \frac{3}{2} \Rightarrow 3\left(\frac{3}{2}\right) + 2B = 0$$

$$2B = \frac{-9}{2}$$

$$B = -\frac{9}{4}$$

$y_p$

$$\underline{y_2 = \frac{3}{2}t - \frac{9}{4}}$$

Check: $y_p' = 3/2$, $y_p'' = 0$

$$y_p'' + 3y_p' + 2y_p = 0 + 3\left(\frac{3}{2}\right) + 2\left(\frac{3}{2}t - \frac{9}{4}\right)$$

$$= \cancel{\frac{9}{4}} + 3t - \cancel{\frac{9}{4}} = 3t \quad \checkmark$$

Strategy #1: If $ay'' + by' + cy = ct^m$ for any $m = 0, 1, 2, 3, \ldots$, we can try

$$y_p(t) = A_m t^m + A_{m-1} t^{m-1} + \cdots + A_1 t + A_0$$

EX: $y'' + 3y' + 2y = 10e^{3t}$

We try $y_p(t) = Ae^{3t}$ $\qquad y_p' = 3Ae^{3t}$ $\qquad y_p'' = 9Ae^{3t}$

$$9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) \overset{\text{set}}{=} 10e^{3t}$$

$$20Ae^{3t} = 10e^{3t}$$

$$\Rightarrow 20A = 10 \qquad \Rightarrow \boxed{y_p = \frac{1}{2}e^{3t}}$$

$$\Rightarrow A = 1/2$$

$y'' + 3y' + 2y = 10e^{-2t}$

Guess: $y = Ae^{-2t}$

$y_p = Ae^{-2t}$

$y_p' = -2Ae^{-2t}$ $\qquad 4Ae^{-2t} + 3(-2Ae^{-2t}) + 2Ae^{-2t} \overset{\text{set}}{=} 10e^{-2t}$

$y_p'' = 4Ae^{-2t}$ $\qquad 0 = 10e^{-2t}$ $\qquad$ uh oh!

Guess better: $y_p = Ate^{-2t}$

$$y_p' = -2Ate^{-2t} + Ae^{-2t}$$

$$y_p'' = 4Ate^{-2t} - 2Ae^{-2t} - 2Ae^{-2t}$$

$$= 4Ate^{-2t} - 4Ae^{-2t}$$

Now Plug in:

$$(\cancel{4Ate^{-2t}} - 4Ae^{-2t}) + 3(\cancel{-2Ate^{-2t}} + Ae^{-2t}) + \cancel{2Ate^{-2t}}$$

$$-4Ae^{-2t} + 3Ae^{-2t} \Rightarrow -Ae^{-2t} \overset{\text{set}}{=} 10e^{-2t} \Rightarrow A = -10 \qquad \boxed{y_p = -10te^{-2t}}$$

Strategy # 2 : If $ay'' + by' + cy = Ce^{kt}$ for any $m = 0, 1, 2, 3, \dots$, we can try

$$y_p(t) = Ae^{kt}$$

Note: If k is a root of the characteristic polynomial, we will have to use

$$y_p(t) = At^m e^{kt}$$

where m is the multiplicity of the root.

EX: $y'' + 3y' + 2y = \sin t$

$1^{st}$ try $y_p(t) = A\sin t$

$y_p' = A\cos t$
$y_p'' = -A\sin t$

$2^{nd}$ try $y_p(t) = A\sin t + B\cos t$

$(-A\sin t - B\cos t) + 3(+A\cos t - B\cos t) + 2(A\sin t + B\cos t)$
$(A - 3B)\sin t + (3A + B)\cos t = \sin t$

$$\left.\begin{aligned} A - 3B &= 1 \\ 3A + B &= 0 \end{aligned}\right\} \rightarrow A = \frac{1}{10} \quad B = \frac{-3}{10}$$

$$y_p = \frac{1}{10}\sin t - \frac{3}{10}\cos t$$

Strategy #3: If $ay'' + by' + cy = C\sin\beta t$ (or $C\cos\beta t$), we can try

$$y_p(t) = A\cos\beta t + B\sin\beta t$$

EX: $y'' + 4y = 5t^2e^t$

Try: $y_p(t) = (At^2 + Bt + C)e^t$

$y_p' = (2At + B)e^t + (At^2 + Bt + C)e^t$
$y_p'' = (2A)e^t + (2At + B)e^t + (2At + B)e^t + (At^2 + Bt + C)e^t$

$$(2A)e^t + (2At + B)e^t + (2At + B)e^t + (At^2 + Bt + C)e^t + 4(At^2 + Bt + C)e^t \overset{set}{=} 5t^2e^t$$

$$5At^2e^t + (4A + 5B)te^t - (2A + 2B + 5C)e^t = 5t^2e^t$$

$5A = 5$
$A = 1$

$4A + 5B = 0$
$4 + 5B = 0$
$B = \frac{-4}{5}$

$2A + 2B + 5C = 0$
$2 + 2\left(\frac{-4}{5}\right) + 5C = 0$
$\frac{2}{5} + 5C = 0$
$C = \frac{-2}{25}$

$$y_p = \left(t^2 - \frac{4}{5}t - \frac{2}{25}\right)e^t$$

GENERAL STRATEGY:

Case 1: No function in the assumed particular solution is a solution of the associated homogeneous differential equation:

| $g(x)$ | Form of $y_p$ |
|---|---|
| **1.** 1 (any constant) | $A$ |
| **2.** $5x+7$ | $Ax+B$ |
| **3.** $3x^2-2$ | $Ax^2+Bx+C$ |
| **4.** $x^3-x+1$ | $Ax^3+Bx^2+Cx+E$ |
| **5.** $\sin 4x$ | $A\cos 4x + B\sin 4x$ |
| **6.** $\cos 4x$ | $A\cos 4x + B\sin 4x$ |
| **7.** $e^{5x}$ | $Ae^{5x}$ |
| **8.** $(9x-2)e^{5x}$ | $(Ax+B)e^{5x}$ |
| **9.** $x^2e^{5x}$ | $(Ax^2+Bx+C)e^{5x}$ |
| **10.** $e^{3x}\sin 4x$ | $Ae^{3x}\cos 4x + Be^{3x}\sin 4x$ |
| **11.** $5x^2\sin 4x$ | $(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$ |
| **12.** $xe^{3x}\cos 4x$ | $(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$ |

Case 2: If any $y_p$ contains terms that duplicate terms in $y_h$ then that $y_p$ term must be multiplied by $x^m$ where $m$ is the smallest positive integer that eliminates that duplication.

EX: Find the FORM of the particular solution to $y''+2y'-3y=g(t)$ where $g(t)$ is given by

$\leftarrow y_h = C_1e^{-3t}+C_2e^{t}$

a) $7\cos 3t$

$y_p = A\cos 3t + B\sin 3t$ ← #6 above

b) $2te^{t}\sin t$

$y_p = (At+B)e^{t}\sin t + (Ct+D)e^{t}\cos t$

c) $t^2\cos \pi t$

$y_p = (At^2+Bt+C)\cos \pi t + (Dt^2+Et+F)\sin \pi t$

d) $5e^{-3t}$

$y_p = Ate^{-3t}$

Because $e^{-3t}$ is a homogenous

e) $3te^{t}$

$y_p = t(At+B)e^{t} = (At^2+Bt)e^{t}$

since $Be^{t}$ is part of homogenous solution

f) $t^2e^{t}$

$y_p = t(At^2+Bt+\underbrace{C)e^{t}}_{\text{homogenous}} = (At^3+Bt^2+Ct)e^{t}$

Note:

1. Don't forget to find a GENERAL solution. So far in this section, we have looked for a particular solution but the final form of the general solution is given by $y = y_h + y_p$

2. This method only works with non-homogeneous that are polynomials, exponentials, sines, cosines, or products of any of these functions.

EX: Solve $y'' + y = 4t + 10\sin t$, $y(\pi)=0$, $y'(\pi)=2$

1) $y''+y=0 \Rightarrow m^2+1=0$

$a \pm bi \rightarrow C_1 e^{at}\sin bt + C_2 e^{at}\cos bt$

$m = \pm i \Rightarrow y_h = C_1 \sin t + C_2 \cos t$

2) Method of Undetermined Coefficients

$y_p = At + B + Ct\sin t + Dt\cos t$

$y_p' = A + Ct\cos t + C\sin t - Dt\sin t + D\cos t$

$y_p'' = -Ct\sin t + C\cos t + C\cos t - Dt\cos t - D\sin t - D\sin t$

$= -Ct\sin t - Dt\cos t + 2C\cos t - 2D\sin t$

$y_p''+y = (-\cancel{Ct\sin t} - \cancel{Dt\cos t} + 2C\cos t - 2D\sin t) + (At + B + \cancel{Ct\sin t} + \cancel{Dt\cos t})$

$= At + B + 2C\cos t - 2D\sin t \overset{\text{set}}{=} 4t + 10\sin t$

$A = 4 \qquad 2C = 0 \Rightarrow C = 0$

$B = 0 \qquad -2D = 10 \Rightarrow D = -5$

$y_p = 4t - 5t\cos t$

$y = y_h + y_p$

$\Rightarrow y = C_1\sin t + C_2\cos t + 4t - 5t\cos t$

I.C.'s $\quad y(\pi)=0, \quad y'(\pi)=2$

$y(\pi) = -C_2 + 4\pi + 5 = 0 \Rightarrow C_2 = 4\pi + 5\pi = 9\pi$

$y' = C_1\cos t - C_2\sin t + 4 + 5t\sin t - 5\cos t$

$y'(\pi) = -C_1 + 4 + 5 = 2 \Rightarrow -C_1 + 9 = 2$

$C_1 = 7$

$$\underline{y = 7\sin t + 9\pi\cos t + 4t - 5t\cos t}$$